Some direct constructions of cyclic $(3,\lambda)$-GDD of type g^v having prescribed number of short orbits

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A (k,λ)-*GDD* of type g^v is an ordered triple $(X,\mathcal{G},\mathcal{B})$, where X is a set of size gv, $\mathcal{G}$ a partition of X into groups of size g, and $\mathcal{B}$ a set of k-subsets of X (called blocks), such that each pair of elements from different groups appears in λ blocks and no block contains two elements from a common group. A GDD is *cyclic* if it admits a cyclic automorphism group G acting sharply transitively on X.

For a cyclic (k,λ)-*GDD* of type g^v, we may assume that $X = Z_{gv}$. Let $B = \{b_1, b_2, \dots, b_k\}$ be a block of a cyclic (k,λ)-GDD of type g^v. The block orbit generated by B is defined as the set of distinct blocks $B + i = \{b_1 + i, b_2 + i, \dots, b_k + i\}$ (mod gv) for $i \in Z_{gv}$. If a block orbit has gv blocks, then the block orbit is said to be *full*, otherwise *short*.

A difference family of an abelian group G is a collection $\{B_1, B_2, \dots, B_t\}$ of k-subsets (called *base blocks*) of G satisfying certain properties. For any base block B of a difference family over an abelian group G, the subgroup

$$\{z \in G : B + z = B\}$$

is called the *stabilizer* of B in G. A base block B is called *full* if its stabilizer is trivial, otherwise it is called *short*. The stabilizer of B is denoted as S_B.

Let H be a subgroup of order h of an abelian group G of order u. A collection $\{B_1, B_2, \dots, B_t\}$ of k-subsets (called *base blocks*) of G forms a (u,h,k,λ) *difference family over G and relative to H with α short blocks* if $\bigcup_{i=1}^{t} \partial B_i$ covers each elements of $G - H$ exactly λ times but no element in H, and there are exactly α short base blocks, where $\partial B = \frac{1}{|S_B|}\{a - b : a, b \in B, a \neq b\}$. We denote such a design as $(u,h,k,\lambda)_\alpha$-DF. When the value of short base blocks is not specified, the design is denoted as (u,h,k,λ)-DF. Observe that if k is a prime and G is cyclic, then we could have short base block only when k is a divisor of u but not of h. For simplicity, our definition is just a special case of difference families. For general information of difference families, the readers refere to [1].

It is not difficult to see that the existence of a $(gv,g,3,\lambda)_\alpha$-DF over Z_{gv} is equivalent to the existence of a cyclic $(3,\lambda)$-GDD of type g^v with α short block orbits. For a cyclic $(3,\lambda)$-GDD of type g^v, the possible short orbit must be generated by $\{0, gv/3, 2gv/3\}$. Therefore in what follows, we only display the full base blocks for a $(gv,g,3,\lambda)_\alpha$-DF over Z_{gv}.

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In the note, $[a,b]$ denotes the set of integers n such that $a \le n \le b$ and λS denotes the multiset containing each element of S exactly λ times for a set S. This note displays some direct constructions of $(u,h,k,\lambda)_\alpha$-DF which are used in [2].

Construction 1 *For $g \equiv 1,5 \pmod 6$ and $g > 1$, there exists a $(3g,g,3,3)_0$-DF over Z_{3g}.*

Proof When $g \equiv 1 \pmod 6$ and $g > 1$, let $g = 6t+1$ where $t \ge 1$. The base blocks are

$2\{0,3t-2,6t-1\}$, $\{0,3t-2,6t+2\}$, $\{0,3t-1,6t+1\}$, $2\{0,3t-1,9t+1\}$, $\{0,3t+1,9t+2\}$,

and when $t \ge 2$, the following base blocks are also used:

$$2\{0,3t-8-6r,6t-4-3r\},\quad 3\{0,3t-7-6r,9t-2-3r\},$$
$$\{0,3t-8-6r,6t-1-3r\},$$

for $r \in [0,t/2-2]$ if t is even ($r \in \emptyset$ when $t=2$), and $r \in [0,(t-3)/2]$ if t is odd,

$$2\{0,3t-4-6r,6t-2-3r\},\quad 3\{0,3t-5-6r,9t-1-3r\},$$
$$\{0,3t-4-6r,6t+1-3r\},$$

for $r \in [0,t/2-1]$ if t is even, and $r \in [0,(t-3)/2]$ if t is odd.

When $g \equiv 5 \pmod 6$, let $g = 6t+5$ where $t \ge 0$. The base blocks are

$2\{0,3t+1,6t+5\}$, $\{0,3t+2,6t+4\}$, $\{0,3t+2,9t+7\}$, $\{0,3t+1,9t+8\}$,

and when $t \ge 1$, the following base blocks are also used:

$$2\{0,3t-5-6r,6t+2-3r\},\quad 2\{0,3t-4-6r,9t+4-3r\},$$
$$\{0,3t-4-6r,6t+1-3r\},\quad \{0,3t-5-6r,9t+5-3r\},$$

for $r \in [0,t/2-1]$ if t is even, and $r \in [0,(t-3)/2]$ if t is odd ($r \in \emptyset$ when $t=1$),

$$2\{0,3t-1-6r,6t+4-3r\},\quad 2\{0,3t-2-6r,9t+5-3r\},$$
$$\{0,3t-2-6r,6t+2-3r\},\quad \{0,3t-1-6r,9t+7-3r\},$$

for $r \in [0,t/2-1]$ if t is even, and $r \in [0,(t-1)/2]$ if t is odd. □

Construction 2 *A $(v,1,3,3)_0$-DF over Z_v exists for $v \equiv 3 \pmod 6$ and $v > 3$.*

Proof The base blocks are

• $v \equiv 3 \pmod{24}$ and $v > 3$: $\{0,1,(v+5)/4\}$, $\{0,(v-3)/8,(v+1)/2\}$,

$\{0,1+2j,(v+5)/4+j\}$, $j \in [0,(v-7)/4]$,

$\{0,2+2j,(v+5)/4+j\}$, $j \in [0,(v-7)/4]\setminus\{(v-3)/8\}$.

• $v \equiv 9 \pmod{24}$: $\{0,(v-1)/8,(v-1)/2\}$, $\{0,(v-1)/4,(v-1)/2\}$,

$\{0,1+2j,(v+3)/4+j\}$, $j \in [0,(v-5)/4]$,

$\{0,2+2j,(v+3)/4+j\}$, $j \in [0,(v-9)/4]\setminus\{(v-9)/8\}$.

• $v \equiv 15 \pmod{24}$: $\{0,(v+1)/8,(v-1)/2\}$, $\{0,(v-3)/4,(v-1)/2\}$,

$\{0,1+2j,(v+1)/4+j\}$, $j \in [0,(v-7)/4]\setminus\{(v-7)/8\}$,

$\{0,2+2j,(v+5)/4+j\}$, $j \in [0,(v-7)/4]$.

• $v \equiv 21 \pmod{24}$: $\{0,1,(v+3)/4\}$, $\{0,(v+3)/8,(v+1)/2\}$,

$\{0,1+2j,(v+3)/4+j\}$, $j \in [0,(v-5)/4]\setminus\{(v-5)/8\}$,

$\{0, 2+2j, (v+7)/4+j\}$, $j \in [0, (v-9)/4]$. □

Construction 3 *There exists a* $(4v, 4, 3, 3)$-PDF *over* Z_{4v} *which is also a* $(2v, 2, 3, 6)_0$-DF *over* Z_{2v} *for* $v \equiv 0 \pmod 3$ *and* $v > 3$.

Proof The base blocks are

- $v \equiv 3 \pmod 6$ and $v > 3$:

$\{0, 1, v-1\}$, $\{0, 1, 2v-1\}$, $\{0, (v-1)/2, (3v+1)/2\}$, $\{0, 2, 6\}$,
$\{0, 1, v+3\}$, $\{0, 2, v+4\}$, $\{0, (v+1)/2, (3v+3)/2\}$,

$\{0, 3+2j, v+2+j\}$, $j \in [0, v-3] \backslash \{(v-3)/2, (v-1)/2\}$,

$\{0, 8+2j, v+5+j\}$, $j \in [0, v-6]$.

- $v \equiv 0 \pmod 6$ and $v \geq 6$:

$\{0, 1, 2\}$, $\{0, 1, 2v-1\}$, $\{0, 2, v+3\}$, $\{0, v/2-1, 3v/2+1\}$, $\{0, v/2+1, 3v/2+2\}$,

$\{0, 3+2j, v+2+j\}$, $j \in [0, v-3] \backslash \{v/2\}$,

$\{0, 4+2j, v+3+j\}$, $j \in [0, v-4] \backslash \{v/2-2\}$. □

Remark: For the definition of perfect difference family (PDF), the readers refer to [2].

Construction 4 *A* $(v, 1, 3, 6)_0$-DF *over* Z_v *exists for* $v \equiv 0 \pmod{12}$.

Proof The base blocks are $\{0, 1, v/4+1\}$, $\{0, 1, v-1\}$, $\{0, 2, v/2+1\}$,

$\{0, 1+2j, v/2+1+j\}$, $j \in [0, v/2-2] \backslash \{v/4-1\}$,

$\{0, 2+2j, v/2+2+j\}$, $j \in [0, v/2-3]$. □

Construction 5 *A* $(v, 1, 3, 12)_0$-DF *over* Z_v *exists for* $v \equiv 6 \pmod{12}$ *and* $v > 6$.

Proof The base blocks are $\{0, 1, 2\}$, $2\{0, 1, v-1\}$, $\{0, 2, v/2+1\}$,

$2\{0, 1+2j, v/2+1+j\}$, $j \in [0, v/2-2]$,

$2\{0, 2+2j, v/2+2+j\}$, $j \in [0, v/2-3]$. □

Construction 6 *For* $g \equiv 2, 4 \pmod 6$ *and* $g > 2$, *there exists a* $(3g, g, 3, 6)_0$-DF *over* Z_{3g}.

Proof When $g \equiv 2 \pmod 6$ and $g > 2$, let $g = 6t+2$ where $t \geq 1$. The base blocks are

$2\{0, 1, 6t-1\}$, $3\{0, 2, 15t+4\}$, $\{0, 6t-2, 18t+2\}$, $2\{0, 6t+1, 18t+2\}$
$3\{0, 1, 9t+2\}$, $\{0, 6t-1, 12t+1\}$, $\{0, 6t+1, 18t+5\}$, $2\{0, 6t+2, 18t+4\}$
$\{0, 2, 12t+4\}$,

and when $t \geq 2$,

$3\{0, 6t-7-6r, 12t-2-3r\}$, $3\{0, 6t-8-6r, 18t-1-3r\}$,
$3\{0, 6t-5-6r, 12t-1-3r\}$, $3\{0, 6t-4-6r, 18t+1-3r\}$,

for $r \in [0, t-2]$.

When $g \equiv 4 \pmod 6$, let $g = 6t+4$ where $t \geq 0$. The base blocks are

$$\begin{array}{llll} \{0,1,6t+2\}, & \{0,2,12t+7\}, & 2\{0,6t+2,18t+10\}, & \{0,6t+4,18t+11\}, \\ \{0,1,12t+8\}, & 2\{0,6t+1,12t+5\}, & & \end{array}$$

and when $t \geq 1$,

$$\begin{array}{ll} 3\{0,6t-5-6r,12t+2-3r\}, & 3\{0,6t-4-6r,18t+7-3r\}, \\ 3\{0,6t-1-6r,12t+4-3r\}, & 3\{0,6t-2-6r,18t+8-3r\}, \end{array}$$

for $r \in [0, t-1]$. □

Construction 7 *For $g \equiv 1, 5 \pmod 6$ and $g > 1$, there exists a $(6g, g, 3, 12)_0$-DF over Z_{6g}.*

Proof When $g \equiv 1 \pmod 6$ and $g > 1$, let $g = 6t+1$ where $t \geq 1$. The base blocks are

$$\begin{array}{llll} 2\{0,1,12t+2\}, & \{0,1,36t+5\}, & 4\{0,12t+1,36t+4\}, & \{0,12t+2,36t+4\}, \\ 2\{0,1,24t+3\}, & & & \end{array}$$

and

$$\begin{array}{ll} 3\{0,12t-11-12r,24t-4-6r\}, & 3\{0,12t-11-12r,24t-3-6r\}, \\ 3\{0,12t-8-12r,24t-3-6r\}, & 3\{0,12t-8-12r,24t-1-6r\}, \\ 3\{0,12t-5-12r,24t-1-6r\}, & 3\{0,12t-5-12r,24t-2-6r\}, \\ 3\{0,12t-4-12r,24t-2-6r\}, & 3\{0,12t-4-12r,24t+1-6r\}, \\ 3\{0,12t-2-12r,24t+1-6r\}, & 3\{0,12t-2-12r,24t+2-6r\}, \\ 6\{0,12t-10-12r,36t-1-6r\}, & 6\{0,12t-3-12r,36t+2-6r\}, \\ 6\{0,12t-9-12r,36t-2-6r\}, & 6\{0,12t-1-12r,36t+3-6r\}, \\ 6\{0,12t-7-12r,36t+1-6r\}, & \end{array}$$

for $r \in [0, t-1]$.

When $g \equiv 5 \pmod 6$, let $g = 6t+5$ where $t \geq 0$. The base blocks are

$$\begin{array}{llll} 2\{0,1,2\}, & 2\{0,12t+8,24t+21\}, & 3\{0,1,18t+17\} & 3\{0,1,30t+26\}, \\ 2\{0,1,12t+9\}, & 2\{0,12t+9,24t+20\}, & 3\{0,2,18t+16\} & 3\{0,2,30t+25\}, \\ 2\{0,2,12t+9\}, & 4\{0,12t+7,24t+20\}, & 3\{0,3,18t+17\} & 3\{0,3,30t+26\}, \\ 2\{0,2,12t+10\}, & 4\{0,12t+10,24t+21\}, & 6\{0,4,18t+19\} & 6\{0,5,30t+27\}, \end{array}$$

and when $t \geq 1$,

$$\begin{array}{ll} 6\{0,12t-4-12r,24t+15-6r\}, & 6\{0,12t-5-12r,36t+22-6r\}, \\ 6\{0,12t-3-12r,24t+14-6r\}, & 6\{0,12t-2-12r,36t+23-6r\}, \\ 6\{0,12t+1-12r,24t+17-6r\}, & 6\{0,12t-1-12r,36t+25-6r\}, \\ 6\{0,12t+2-12r,24t+16-6r\}, & 6\{0,12t+3-12r,36t+26-6r\}, \\ 6\{0,12t+4-12r,24t+19-6r\}, & 6\{0,12t+5-12r,36t+27-6r\}, \end{array}$$

for $r \in [0, t-1]$. □

Construction 8 *A $(2v, 2, 3, 3)_3$-DF over Z_{2v} exists for $v \equiv 9 \pmod{12}$.*

Proof The base blocks are

$\{0,1,v/3+1\}$, $\{0,(v+3)/6,(v-1)/2\}$, $\{0,v/3,(5v+3)/6\}$, $\{0,(v+3)/4,v+1\}$,

$\{0,1+2j,(v+1)/2+j\}$, $j \in [0,(v-3)/2] \setminus \{(v-3)/6\}$,

$\{0,2+2j,(v+3)/2+j\}$, $j \in [0,(v-5)/2] \setminus \{(v-9)/6,(v-5)/4,v/3-1\}$. □

Construction 9 *A* $(8v, 8, 3, 3)_3$-DF *over* Z_{8v} *exists for* $v \equiv 3 \pmod 6$ *and* $v > 3$.

Proof The base blocks are

$$\begin{array}{llll}
\{0,1,4v/3\}, & \{0,4,2v+3\}, & \{0,4v/3-1,10v/3\}, & \{0,2v-2,4v-1\}, \\
\{0,1,3v+2\}, & \{0,(v+1)/2,2v+1\}, & \{0,4v/3,8v/3+1\}, & \{0,2v-1,(9v-1)/2\}, \\
\{0,2,v+1\}, & \{0,2v/3,2v+2\}, & &
\end{array}$$

$\{0, 6+2j, 2v+3+j\}$, $j \in [0, 2v-4] \backslash \{2v/3-3, 2v/3-2, v-3, 4v/3-3\}$,

$\{0, 3+2j, 2v+2+j\}$, $j \in [0, 2v-3] \backslash \{(v-3)/2, 2v/3-2, v-2, v-1, v, (3v-3)/2\}$. □

Construction 10 *For* $g \equiv 5 \pmod 6$, *there exists a* $(3g, g, 3, 3)_3$-DF *over* Z_{3g}.

Proof Let $g = 6t+5$ where $t \geq 0$. The base blocks are

$$\{0,3t+1,6t+2\},\quad \{0,3t+2,6t+4\},\quad \{0,3t+1,9t+8\},\quad \{0,6t+4,12t+8\},$$

and when $t \geq 1$,

$$\begin{array}{ll}
2\{0,3t-2-6r,6t+2-3r\}, & 2\{0,3t-1-6r,9t+7-3r\}, \\
\{0,3t-1-6r,6t+1-3r\}, & \{0,3t-2-6r,9t+5-3r\},
\end{array}$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-1)/2]$ if t is odd,

$$\begin{array}{ll}
2\{0,3t-4-6r,6t+1-3r\}, & 2\{0,3t-5-6r,9t+5-3r\}, \\
\{0,3t-5-6r,6t-1-3r\}, & \{0,3t-4-6r,9t+4-3r\},
\end{array}$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd ($r \in \emptyset$ when $t = 1$). □

Construction 11 *For* $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 3$ *and* $6 \leq v \leq 21$, *there exists a* $(gv, g, 3, 3)_3$-DF *over* Z_{gv}.

Proof Let $g = 12t+8$ where $t \geq 0$.

- $v = 6$: The base blocks are

$$\begin{array}{llll}
2\{0,1,4\}, & 2\{0,2,30t+19\}, & \{0,12t+7,30t+21\}, & 2\{0,12t+8,30t+21\}, \\
\{0,1,5\}, & \{0,5,30t+22\}, & \{0,12t+7,30t+22\}, & 2\{0,12t+9,30t+20\}, \\
\{0,2,5\}, & \{0,12t+7,30t+20\}, & \{0,12t+8,30t+22\}, & 2\{0,18t+10,36t+25\}, \\
 & & \{0,12t+9,30t+19\}, & \{0,18t+11,36t+25\},
\end{array}$$

and when $t \geq 1$,

$$\begin{array}{ll}
3\{0,12t-5-12r,24t+10-6r\}, & 3\{0,12t-4-12r,36t+17-6r\}, \\
3\{0,12t-2-12r,24t+11-6r\}, & 3\{0,12t-3-12r,36t+19-6r\}, \\
3\{0,12t-1-12r,24t+13-6r\}, & 3\{0,12t+1-12r,36t+20-6r\}, \\
3\{0,12t+3-12r,24t+14-6r\}, & 3\{0,12t+2-12r,36t+22-6r\}, \\
3\{0,12t+5-12r,24t+15-6r\}, & 3\{0,12t+4-12r,36t+21-6r\},
\end{array}$$

for $r \in [0, t-1]$.

- $v = 9$: The base blocks are

$$\begin{array}{llll}
\{0,4,8\}, & \{0,2,45t+32\}, & 2\{0,3,45t+32\}, & 3\{0,18t+10,36t+22\}, \\
3\{0,1,27t+17\}, & \{0,3,45t+28\}, & 2\{0,8,45t+33\}, & 3\{0,18t+11,54t+34\}, \\
3\{0,6,27t+19\}, & \{0,4,45t+33\}, & 3\{0,5,45t+31\}, & 3\{0,27t+15,54t+35\}, \\
3\{0,7,27t+21\}, & 2\{0,2,45t+30\}, & &
\end{array}$$

and when $t \geq 1$,

$$\begin{array}{ll}
3\{0, 18t-8-18r, 36t+13-9r\}, & 3\{0, 18t-7-18r, 54t+26-9r\},\\
3\{0, 18t-5-18r, 36t+15-9r\}, & 3\{0, 18t-6-18r, 54t+25-9r\},\\
3\{0, 18t-3-18r, 36t+14-9r\}, & 3\{0, 18t-4-18r, 54t+28-9r\},\\
3\{0, 18t-2-18r, 36t+17-9r\}, & 3\{0, 18t-1-18r, 54t+29-9r\},\\
3\{0, 18t+1-18r, 36t+16-9r\}, & 3\{0, 18t+2-18r, 54t+31-9r\},\\
3\{0, 18t+3-18r, 36t+19-9r\}, & 3\{0, 18t+4-18r, 54t+30-9r\},\\
3\{0, 18t+6-18r, 36t+20-9r\}, & 3\{0, 18t+5-18r, 54t+33-9r\},\\
3\{0, 18t+8-18r, 36t+21-9r\}, & 3\{0, 18t+7-18r, 54t+32-9r\},
\end{array}$$

for $r \in [0, t-1]$.

- $v = 12$: The base blocks are

$$\begin{array}{llll}
\{0,1,10\}, & 3\{0,3,36t+31\}, & \{0,6,60t+40\}, & 3\{0,24t+16,60t+38\},\\
\{0,2,8\}, & 2\{0,2,60t+39\}, & \{0,7,60t+44\}, & 3\{0,24t+17,60t+43\},\\
\{0,5,11\}, & 2\{0,5,60t+40\}, & 3\{0,24t+14,48t+33\}, & 3\{0,24t+18,60t+41\},\\
2\{0,1,9\}, & 2\{0,10,60t+44\}, & 3\{0,24t+13,60t+42\}, & 3\{0,36t+20,72t+47\},\\
2\{0,4,11\}, & \{0,4,60t+39\}, & 3\{0,24t+15,60t+45\}, & 3\{0,36t+21,72t+46\},
\end{array}$$

and when $t \geq 1$,

$$\begin{array}{ll}
3\{0, 24t-10-24r, 48t+20-12r\}, & 3\{0, 24t-11-24r, 72t+34-12r\},\\
3\{0, 24t-9-24r, 48t+22-12r\}, & 3\{0, 24t-7-24r, 72t+37-12r\},\\
3\{0, 24t-8-24r, 48t+21-12r\}, & 3\{0, 24t-6-24r, 72t+35-12r\},\\
3\{0, 24t-4-24r, 48t+23-12r\}, & 3\{0, 24t-5-24r, 72t+38-12r\},\\
3\{0, 24t-2-24r, 48t+26-12r\}, & 3\{0, 24t-3-24r, 72t+39-12r\},\\
3\{0, 24t-1-24r, 48t+25-12r\}, & 3\{0, 24t+1-24r, 72t+41-12r\},\\
3\{0, 24t+2-24r, 48t+27-12r\}, & 3\{0, 24t+3-24r, 72t+40-12r\},\\
3\{0, 24t+5-24r, 48t+28-12r\}, & 3\{0, 24t+4-24r, 72t+43-12r\},\\
3\{0, 24t+8-24r, 48t+29-12r\}, & 3\{0, 24t+6-24r, 72t+44-12r\},\\
3\{0, 24t+9-24r, 48t+31-12r\}, & 3\{0, 24t+7-24r, 72t+42-12r\},\\
3\{0, 24t+10-24r, 48t+30-12r\}, & 3\{0, 24t+11-24r, 72t+45-12r\},
\end{array}$$

for $r \in [0, t-1]$.

- $v = 15$: The base blocks are

$$\begin{array}{llll}
\{0,6,12\}, & 3\{0,14,45t+36\}, & 2\{0,3,75t+47\}, & 3\{0,30t+18,60t+37\},\\
3\{0,5,30t+21\}, & \{0,2,75t+44\}, & 2\{0,8,75t+50\}, & 3\{0,30t+17,90t+56\},\\
3\{0,1,45t+29\}, & \{0,3,75t+50\}, & 2\{0,12,75t+55\}, & 3\{0,30t+20,90t+58\},\\
3\{0,9,45t+34\}, & \{0,6,75t+55\}, & 3\{0,4,75t+52\}, & 3\{0,45t+26,90t+57\},\\
3\{0,10,45t+33\}, & \{0,8,75t+51\}, & 3\{0,7,75t+53\}, & 3\{0,45t+27,90t+59\},\\
3\{0,11,45t+35\}, & 2\{0,2,75t+51\}, & 3\{0,13,75t+54\}, &
\end{array}$$

and when $t \geq 1$,

$3\{0,30t-14-30r,60t+22-15r\}$, $3\{0,30t-13-30r,90t+42-15r\}$,
$3\{0,30t-10-30r,60t+25-15r\}$, $3\{0,30t-12-30r,90t+41-15r\}$,
$3\{0,30t-9-30r,60t+23-15r\}$, $3\{0,30t-11-30r,90t+43-15r\}$,
$3\{0,30t-8-30r,60t+26-15r\}$, $3\{0,30t-5-30r,90t+47-15r\}$,
$3\{0,30t-7-30r,60t+24-15r\}$, $3\{0,30t-4-30r,90t+44-15r\}$,
$3\{0,30t-6-30r,60t+27-15r\}$, $3\{0,30t-3-30r,90t+46-15r\}$,
$3\{0,30t+1-30r,60t+28-15r\}$, $3\{0,30t-2-30r,90t+48-15r\}$,
$3\{0,30t+2-30r,60t+31-15r\}$, $3\{0,30t-1-30r,90t+50-15r\}$,
$3\{0,30t+3-30r,60t+29-15r\}$, $3\{0,30t+5-30r,90t+49-15r\}$,
$3\{0,30t+4-30r,60t+32-15r\}$, $3\{0,30t+6-30r,90t+52-15r\}$,
$3\{0,30t+9-30r,60t+33-15r\}$, $3\{0,30t+7-30r,90t+54-15r\}$,
$3\{0,30t+10-30r,60t+35-15r\}$, $3\{0,30t+8-30r,90t+51-15r\}$,
$3\{0,30t+11-30r,60t+34-15r\}$, $3\{0,30t+12-30r,90t+53-15r\}$,
$3\{0,30t+14-30r,60t+36-15r\}$, $3\{0,30t+13-30r,90t+55-15r\}$,

for $r \in [0,t-1]$.

- $v = 18$: The base blocks are

$\{0,4,12\}$, $3\{0,10,54t+45\}$, $3\{0,36t+20,72t+49\}$, $3\{0,36t+25,90t+59\}$,
$\{0,8,16\}$, $3\{0,2,90t+58\}$, $3\{0,36t+23,72t+50\}$, $3\{0,36t+26,90t+64\}$,
$2\{0,4,16\}$, $3\{0,5,90t+62\}$, $3\{0,36t+19,90t+65\}$, $3\{0,36t+28,90t+68\}$,
$3\{0,1,14\}$, $3\{0,7,90t+60\}$, $3\{0,36t+21,90t+63\}$, $3\{0,54t+30,108t+71\}$,
$3\{0,6,17\}$, $3\{0,9,90t+61\}$, $3\{0,36t+22,90t+55\}$, $3\{0,54t+31,108t+70\}$,
$3\{0,3,54t+47\}$, $3\{0,15,90t+66\}$, $3\{0,36t+24,90t+67\}$, $3\{0,54t+32,108t+69\}$,

and when $t \geq 1$,

$3\{0,36t-17-36r,72t+30-18r\}$, $3\{0,36t-16-36r,108t+51-18r\}$,
$3\{0,36t-15-36r,72t+31-18r\}$, $3\{0,36t-14-36r,108t+52-18r\}$,
$3\{0,36t-11-36r,72t+34-18r\}$, $3\{0,36t-13-36r,108t+55-18r\}$,
$3\{0,36t-10-36r,72t+32-18r\}$, $3\{0,36t-12-36r,108t+53-18r\}$,
$3\{0,36t-9-36r,72t+35-18r\}$, $3\{0,36t-7-36r,108t+56-18r\}$,
$3\{0,36t-8-36r,72t+33-18r\}$, $3\{0,36t-6-36r,108t+58-18r\}$,
$3\{0,36t-4-36r,72t+39-18r\}$, $3\{0,36t-5-36r,108t+57-18r\}$,
$3\{0,36t-3-36r,72t+37-18r\}$, $3\{0,36t-2-36r,108t+59-18r\}$,
$3\{0,36t-1-36r,72t+38-18r\}$, $3\{0,36t+1-36r,108t+61-18r\}$,
$3\{0,36t+2-36r,72t+40-18r\}$, $3\{0,36t+3-36r,108t+62-18r\}$,
$3\{0,36t+6-36r,72t+41-18r\}$, $3\{0,36t+4-36r,108t+60-18r\}$,
$3\{0,36t+7-36r,72t+44-18r\}$, $3\{0,36t+5-36r,108t+63-18r\}$,
$3\{0,36t+10-36r,72t+43-18r\}$, $3\{0,36t+8-36r,108t+65-18r\}$,
$3\{0,36t+11-36r,72t+45-18r\}$, $3\{0,36t+9-36r,108t+64-18r\}$,
$3\{0,36t+12-36r,72t+42-18r\}$, $3\{0,36t+13-36r,108t+66-18r\}$,
$3\{0,36t+14-36r,72t+46-18r\}$, $3\{0,36t+15-36r,108t+67-18r\}$,
$3\{0,36t+16-36r,72t+47-18r\}$, $3\{0,36t+17-36r,108t+68-18r\}$,

for $r \in [0,t-1]$.

- $v = 21$: The base blocks are

$\{0,10,20\}$, $3\{0,19,63t+51\}$, $2\{0,2,105t+71\}$, $3\{0,16,105t+74\}$,
$3\{0,6,42t+30\}$, $\{0,1,105t+70\}$, $2\{0,3,105t+67\}$, $3\{0,42t+25,84t+52\}$,
$3\{0,7,42t+29\}$, $\{0,2,105t+64\}$, $2\{0,4,105t+70\}$, $3\{0,42t+23,126t+78\}$,
$3\{0,8,63t+46\}$, $\{0,3,105t+71\}$, $2\{0,5,105t+73\}$, $3\{0,42t+26,126t+79\}$,
$3\{0,9,63t+45\}$, $\{0,4,105t+77\}$, $2\{0,20,105t+77\}$, $3\{0,42t+28,126t+82\}$,
$3\{0,12,63t+47\}$, $\{0,5,105t+66\}$, $3\{0,11,105t+76\}$, $3\{0,63t+37,126t+81\}$,
$3\{0,14,63t+48\}$, $\{0,10,105t+67\}$, $3\{0,13,105t+72\}$, $3\{0,63t+39,126t+80\}$,
$3\{0,17,63t+50\}$, $2\{0,1,105t+62\}$, $3\{0,15,105t+75\}$, $3\{0,63t+40,126t+83\}$,
$3\{0,18,63t+49\}$,

and when $t \geq 1$,

$3\{0,42t-19-42r,84t+31-21r\}$, $3\{0,42t-20-42r,126t+57-21r\}$,
$3\{0,42t-17-42r,84t+34-21r\}$, $3\{0,42t-18-42r,126t+58-21r\}$,
$3\{0,42t-15-42r,84t+32-21r\}$, $3\{0,42t-16-42r,126t+59-21r\}$,
$3\{0,42t-13-42r,84t+33-21r\}$, $3\{0,42t-14-42r,126t+60-21r\}$,
$3\{0,42t-12-42r,84t+37-21r\}$, $3\{0,42t-11-42r,126t+61-21r\}$,
$3\{0,42t-10-42r,84t+38-21r\}$, $3\{0,42t-8-42r,126t+62-21r\}$,
$3\{0,42t-9-42r,84t+35-21r\}$, $3\{0,42t-7-42r,126t+66-21r\}$,
$3\{0,42t-6-42r,84t+39-21r\}$, $3\{0,42t-4-42r,126t+65-21r\}$,
$3\{0,42t-5-42r,84t+36-21r\}$, $3\{0,42t-3-42r,126t+68-21r\}$,
$3\{0,42t+1-42r,84t+40-21r\}$, $3\{0,42t-2-42r,126t+64-21r\}$,
$3\{0,42t+3-42r,84t+46-21r\}$, $3\{0,42t-1-42r,126t+67-21r\}$,
$3\{0,42t+4-42r,84t+44-21r\}$, $3\{0,42t+2-42r,126t+69-21r\}$,
$3\{0,42t+5-42r,84t+41-21r\}$, $3\{0,42t+6-42r,126t+71-21r\}$,
$3\{0,42t+7-42r,84t+45-21r\}$, $3\{0,42t+9-42r,126t+73-21r\}$,
$3\{0,42t+8-42r,84t+43-21r\}$, $3\{0,42t+10-42r,126t+70-21r\}$,
$3\{0,42t+13-42r,84t+50-21r\}$, $3\{0,42+11-42r,126t+72-21r\}$,
$3\{0,42t+14-42r,84t+47-21r\}$, $3\{0,42t+12-42r,126t+74-21r\}$,
$3\{0,42t+15-42r,84t+49-21r\}$, $3\{0,42t+17-42r,126t+75-21r\}$,
$3\{0,42t+16-42r,84t+48-21r\}$, $3\{0,42t+18-42r,126t+77-21r\}$,
$3\{0,42t+20-42r,84t+51-21r\}$, $3\{0,42t+19-42r,126t+76-21r\}$,

for $r \in [0, t-1]$. □

Construction 12 *A* $(v,1,3,4)_1$-DF *over* Z_v *exists for* $v \equiv 0 \pmod 3$ *and* $v > 9$.

Proof For $v \equiv 3 \pmod 6$ and $v > 9$, taking together the base blocks of a $(v,1,3,1)_1$-DF over Z_v from Lemma 4.4(1) and a $(v,1,3,3)_0$-DF over Z_v from Lemma 3.5 both in [2], we can get the desired $(v,1,3,4)_1$-DF over Z_v. For $v \equiv 0 \pmod 6$ and $v > 9$, the base blocks are

- $v \equiv 6 \pmod{12}$ and $v \geq 18$: $\{0, v/6, v/3\}$, $2\{0,(v-2)/4, v/2\}$,

$4\{0, 1+2j, (5v+6)/12+j\}$, $j \in [0,(v-18)/12]$,

$2\{0, 2+2j, (v+6)/4+j\}$, $j \in [0,(v-18)/12]$,

$2\{0, 2+2j, (v+2)/4+j\}$, $j \in [0,(v-18)/12]$.

- $v \equiv 0 \pmod{12}$ and $v \geq 12$:

$\{0, v/6, v/3\}$, $2\{0,1,v/3\}$, $2\{0, v/4-1, v/2-1\}$, $2\{0, v/6-1, 7v/12-1\}$,

$2\{0, 1+2j, 5v/12+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v=12$),

$2\{0, 3+2j, 5v/12+2+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v=12$),

$2\{0, 2+2j, v/4+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v=12$),

$2\{0,2+2j,v/4+1+j\}$, $j\in[0,v/12-2]$ ($j\in\emptyset$ if $v=12$). □

Construction 13 *A* $(2v,2,3,6)_6$-DF *over* Z_{2v} *exists for* $v\equiv 0 \pmod 3$ *and* $v\geq 9$.

Proof For $v=9,12$, repeating the base blocks of a $(2v,2,3,3)_3$-DF over Z_{2v} from Lemma 4.5 of [2] twice to get the conclusion. For $v\equiv 0 \pmod 3$ and $v\geq 15$, the base blocks are

- $v\equiv 0 \pmod 6$ and $v\geq 24$:

$\{0,1,2\}$, $5\{0,2,v/2\}$, $3\{0,v/3-2,2v/3-2\}$, $2\{0,v/3-2,7v/6-1\}$,
$\{0,1,v/3-2\}$, $\{0,3,2v/3+2\}$, $\{0,v/3-1,2v/3-2\}$, $\{0,v/3-1,7v/6-2\}$,
$\{0,1,5v/6+1\}$, $5\{0,3,5v/6+2\}$, $3\{0,v/3-1,2v/3-1\}$, $\{0,v/2-2,7v/6-1\}$,
$2\{0,1,2v/3-1\}$, $5\{0,v/3-3,v-1\}$, $\{0,v/2-1,v-1\}$, $5\{0,v/2-1,7v/6\}$,

$6\{0,5+2j,5v/6+3+j\}$, $j\in[0,v/6-5]$ ($j\in\emptyset$ if $v=24$),

$6\{0,4+2j,v/2+1+j\}$, $j\in[0,v/6-4]$.

- $v\equiv 3 \pmod 6$ and $v\geq 21$:

$\{0,1,v/3-1\}$, $5\{0,2,(v+1)/2\}$, $5\{0,v/3-1,2v/3-1\}$, $\{0,v/3,(7v-3)/6\}$,
$\{0,1,2v/3+2\}$, $5\{0,v/3-4,v-2\}$, $\{0,(v-3)/2,v-2\}$, $\{0,(v-1)/2,(7v+3)/6\}$,
$4\{0,1,(5v+3)/6\}$, $4\{0,v/3-2,v-1\}$, $2\{0,(v-1)/2,v-1\}$, $\{0,(v+1)/2,(7v-3)/6\}$,
$\{0,2,v/3-2\}$,

$6\{0,3+2j,(5v+9)/6+j\}$, $j\in[0,(v-27)/6]$ ($j\in\emptyset$ if $v=21$),

$6\{0,4+2j,(v+3)/2+j\}$, $j\in[0,(v-21)/6]$.

- $v=18$:

$\{0,2,8\}$, $\{0,2,13\}$, $\{0,4,13\}$, $2\{0,1,6\}$, $2\{0,6,16\}$, $4\{0,3,17\}$, $4\{0,5,13\}$,
$\{0,2,10\}$, $\{0,4,10\}$, $2\{0,1,3\}$, $2\{0,1,15\}$, $2\{0,7,17\}$, $4\{0,4,15\}$, $4\{0,7,16\}$,
$\{0,2,11\}$.

- $v=15$:

$\{0,2,7\}$, $\{0,5,13\}$, $2\{0,1,4\}$, $2\{0,3,14\}$, $4\{0,2,9\}$, $4\{0,4,12\}$, $4\{0,5,11\}$,
$\{0,2,8\}$, $\{0,6,13\}$, $2\{0,3,12\}$, $4\{0,1,14\}$. □

Construction 14 *For* $g\equiv 4 \pmod 6$, *there exists a* $(3g,g,3,4)_1$-DF *over* Z_{3g}.

Proof Let $g=6t+4$ where $t\geq 0$. The base blocks are

$$\{0,3t+1,6t+2\},\quad \{0,3t+2,6t+4\},\quad 2\{0,3t+1,9t+5\},\quad \{0,3t+2,9t+7\},$$

and when $t\geq 1$,

$$\{0,3t-1-6r,6t+1-3r\},\quad \{0,3t-2-6r,9t+5-3r\},$$
$$3\{0,3t-2-6r,6t+2-3r\},\quad 3\{0,3t-1-6r,9t+4-3r\},$$

for $r\in[0,t/2-1]$ if t is even, and $r\in[0,(t-1)/2]$ if t is odd,

$$\{0,3t-5-6r,6t-1-3r\},\quad \{0,3t-4-6r,9t+4-3r\},$$
$$3\{0,3t-4-6r,6t+1-3r\},\quad 3\{0,3t-5-6r,9t+2-3r\},$$

for $r\in[0,t/2-1]$ if t is even, and $r\in[0,(t-3)/2]$ if t is odd ($r\in\emptyset$ when $t=1$). □

Construction 15 *For* $g\equiv 2 \pmod 6$ *and* $g>2$, *there exists a* $(3g,g,3,6)_6$-DF *over* Z_{3g}.

Proof Let $g = 6t + 2$ where $t \geq 1$. The base blocks are

$$2\{0, 3t-2, 6t-1\},\quad 2\{0, 3t-1, 6t+1\},\quad \{0, 3t+2, 6t+4\},\quad \{0, 6t+1, 12t+2\},$$
$$2\{0, 3t-1, 6t-2\},\quad 2\{0, 3t+1, 6t+5\},\quad 4\{0, 3t-2, 9t+2\},$$

and when $t \geq 2$,

$$2\{0, 3t-4-6r, 6t+1-3r\},\quad 2\{0, 3t-5-6r, 9t+2-3r\},$$
$$2\{0, 3t-5-6r, 6t-4-3r\},\quad 4\{0, 3t-4-6r, 9t+1-3r\},$$
$$2\{0, 3t-5-6r, 6t-1-3r\},$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

$$2\{0, 3t-8-6r, 6t-1-3r\},\quad 2\{0, 3t-7-6r, 9t+1-3r\},$$
$$2\{0, 3t-7-6r, 6t-5-3r\},\quad 4\{0, 3t-8-6r, 9t-1-3r\},$$
$$2\{0, 3t-7-6r, 6t-2-3r\},$$

for $r \in [0, t/2-2]$ if t is even ($r \in \emptyset$ when $t = 2$), and $r \in [0, (t-3)/2]$ if t is odd. □

Construction 16 *For* $g \equiv 10 \pmod{12}$, *there exists a* $(6g, g, 3, 4)_1$-DF *over* Z_{6g}.

Proof Let $g = 12t + 10$ where $t \geq 0$. The base blocks are

$$2\{0, 1, 18t+14\},\quad 2\{0, 5, 18t+16\},\quad 2\{0, 4, 30t+27\},\quad 3\{0, 12t+9, 36t+28\},$$
$$2\{0, 2, 18t+15\},\quad 2\{0, 1, 30t+26\},\quad 2\{0, 5, 30t+27\},\quad \{0, 12t+8, 36t+28\},$$
$$2\{0, 2, 18t+16\},\quad 2\{0, 3, 30t+25\},\quad 4\{0, 12t+7, 24t+17\},\quad \{0, 12t+9, 36t+29\},$$
$$2\{0, 4, 18t+15\},\quad 2\{0, 3, 30t+26\},\quad 3\{0, 12t+8, 36t+29\},\quad \{0, 24t+19, 48t+39\},$$

and when $t \geq 1$,

$$4\{0, 12t-5-12r, 24t+11-6r\},\quad 4\{0, 12t-4-12r, 36t+23-6r\},$$
$$4\{0, 12t-2-12r, 24t+13-6r\},\quad 4\{0, 12t-3-12r, 36t+22-6r\},$$
$$4\{0, 12t+1-12r, 24t+14-6r\},\quad 4\{0, 12t-1-12r, 36t+25-6r\},$$
$$4\{0, 12t+2-12r, 24t+16-6r\},\quad 4\{0, 12t+3-12r, 36t+26-6r\},$$
$$4\{0, 12t+4-12r, 24t+15-6r\},\quad 4\{0, 12t+5-12r, 36t+27-6r\},$$

for $r \in [0, t-1]$. □

Construction 17 *For* $g \equiv 5 \pmod 6$, $\alpha \in \{1, 7\}$, *there exists a* $(6g, g, 3, 8)_\alpha$-DF *over* Z_{6g}.

Proof Let $g = 6t + 5$ where $t \geq 0$.

- $\alpha = 1$: The base blocks are

$$\{0, 6t+2, 12t+9\},\quad \{0, 6t+4, 12t+11\},\quad 3\{0, 6t+3, 12t+7\},\quad 3\{0, 6t+2, 18t+11\},$$
$$\{0, 6t+3, 12t+8\},\quad 2\{0, 6t+2, 12t+5\},\quad \{0, 6t+4, 18t+11\},\quad 4\{0, 6t+1, 18t+14\},$$
$$\{0, 6t+4, 12t+8\},\quad 2\{0, 6t+2, 12t+10\},\quad 2\{0, 6t+5, 18t+13\},\quad 4\{0, 6t+1, 18t+15\},$$
$$\{0, 6t+4, 12t+9\},\quad 2\{0, 6t+3, 12t+10\},\quad 2\{0, 6t+8, 18t+13\},\quad 3\{0, 12t+9, 24t+19\},$$

and when $t \geq 2$,

$$
\begin{array}{ll}
4\{0, 6t-11-12r, 12t+3-6r\}, & 4\{0, 6t-10-12r, 18t+5-6r\}, \\
4\{0, 6t-8-12r, 12t+5-6r\}, & 4\{0, 6t-9-12r, 18t+7-6r\}, \\
4\{0, 6t-7-12r, 12t+4-6r\}, & 4\{0, 6t-5-12r, 18t+8-6r\}, \\
4\{0, 6t-3-12r, 12t+7-6r\}, & 4\{0, 6t-4-12r, 18t+10-6r\}, \\
4\{0, 6t-1-12r, 12t+8-6r\}, & 4\{0, 6t-2-12r, 18t+9-6r\}, \\
4\{0, 6t-11-12r, 12t-1-6r\}, & 4\{0, 6t-10-12r, 18t+9-6r\}, \\
4\{0, 6t-8-12r, 12t+1-6r\}, & 4\{0, 6t-9-12r, 18t+11-6r\}, \\
4\{0, 6t-5-12r, 12t+2-6r\}, & 4\{0, 6t-7-12r, 18t+10-6r\}, \\
4\{0, 6t-4-12r, 12t+4-6r\}, & 4\{0, 6t-3-12r, 18t+13-6r\}, \\
4\{0, 6t-2-12r, 12t+3-6r\}, & 4\{0, 6t-1-12r, 18t+14-6r\},
\end{array}
$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is odd,

$$
\begin{array}{lllll}
4\{0, 2, 9t+6\}, & 4\{0, 3, 9t+10\}, & 4\{0, 5, 9t+11\}, & 4\{0, 1, 15t+17\}, & 4\{0, 4, 15t+12\}, \\
4\{0, 3, 9t+8\}, & 4\{0, 5, 9t+7\}, & 4\{0, 1, 15t+13\}, & 4\{0, 2, 15t+13\}, & 4\{0, 4, 15t+14\}.
\end{array}
$$

- $\alpha = 7$: The base blocks are

$$
\begin{array}{llll}
\{0, 6t+2, 12t+9\}, & \{0, 6t+4, 12t+11\}, & 3\{0, 6t+3, 12t+7\}, & 3\{0, 6t+2, 18t+11\}, \\
\{0, 6t+3, 12t+8\}, & 2\{0, 6t+2, 12t+5\}, & \{0, 6t+4, 18t+11\}, & 4\{0, 6t+1, 18t+14\}, \\
\{0, 6t+4, 12t+8\}, & 2\{0, 6t+2, 12t+9\}, & 2\{0, 6t+5, 18t+13\}, & 4\{0, 6t+1, 18t+15\}, \\
\{0, 6t+4, 12t+9\}, & 2\{0, 6t+3, 12t+11\}, & 2\{0, 6t+8, 18t+13\}, & \{0, 12t+9, 24t+19\},
\end{array}
$$

and when $t \geq 2$,

$$
\begin{array}{ll}
4\{0, 6t-11-12r, 12t+3-6r\}, & 4\{0, 6t-10-12r, 18t+5-6r\}, \\
4\{0, 6t-8-12r, 12t+5-6r\}, & 4\{0, 6t-9-12r, 18t+7-6r\}, \\
4\{0, 6t-7-12r, 12t+4-6r\}, & 4\{0, 6t-5-12r, 18t+8-6r\}, \\
4\{0, 6t-3-12r, 12t+7-6r\}, & 4\{0, 6t-4-12r, 18t+10-6r\}, \\
4\{0, 6t-1-12r, 12t+8-6r\}, & 4\{0, 6t-2-12r, 18t+9-6r\}, \\
4\{0, 6t-11-12r, 12t-1-6r\}, & 4\{0, 6t-10-12r, 18t+9-6r\}, \\
4\{0, 6t-8-12r, 12t+1-6r\}, & 4\{0, 6t-9-12r, 18t+11-6r\}, \\
4\{0, 6t-5-12r, 12t+2-6r\}, & 4\{0, 6t-7-12r, 18t+10-6r\}, \\
4\{0, 6t-4-12r, 12t+4-6r\}, & 4\{0, 6t-3-12r, 18t+13-6r\}, \\
4\{0, 6t-2-12r, 12t+3-6r\}, & 4\{0, 6t-1-12r, 18t+14-6r\},
\end{array}
$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is odd,

$$
\begin{array}{lllll}
4\{0, 2, 9t+6\}, & 4\{0, 3, 9t+10\}, & 4\{0, 5, 9t+11\}, & 4\{0, 1, 15t+17\}, & 4\{0, 4, 15t+12\}, \\
4\{0, 3, 9t+8\}, & 4\{0, 5, 9t+7\}, & 4\{0, 1, 15t+13\}, & 4\{0, 2, 15t+13\}, & 4\{0, 4, 15t+14\}.
\end{array}
$$

□

Construction 18 *For $g \equiv 1 \pmod 6$ and $g > 1$, there exists a $(6g, g, 3, 4)_4$-DF over Z_{6g}.*

Proof Let $g = 6t+1$ where $t \geq 1$. The base blocks are

$$
\begin{array}{llll}
\{0, 6t-5, 12t-1\}, & 2\{0, 6t-3, 12t-1\}, & \{0, 6t-5, 18t+2\}, & 2\{0, 6t-4, 18t-1\}, \\
\{0, 6t-2, 12t-1\}, & 2\{0, 6t-2, 12t+1\}, & \{0, 6t-2, 18t+2\}, & 2\{0, 6t-4, 18t+1\}, \\
\{0, 6t+1, 12t+4\}, & 2\{0, 6t-1, 12t-2\}, & 2\{0, 6t-5, 18t+3\}, & 2\{0, 6t-3, 18t-2\}, \\
\{0, 6t+3, 12t+7\}, & 2\{0, 6t+1, 12t+3\}, & &
\end{array}
$$

and when $t \geq 3$,

$$
\begin{array}{ll}
2\{0, 6t-16-12r, 12t-7-6r\}, & 2\{0, 6t-17-12r, 18t-3-6r\}, \\
2\{0, 6t-15-12r, 12t-5-6r\}, & 2\{0, 6t-14-12r, 18t-1-6r\}, \\
2\{0, 6t-11-12r, 12t-4-6r\}, & 2\{0, 6t-13-12r, 18t-2-6r\}, \\
2\{0, 6t-10-12r, 12t-2-6r\}, & 2\{0, 6t-9-12r, 18t+1-6r\}, \\
2\{0, 6t-8-12r, 12t-3-6r\}, & 2\{0, 6t-7-12r, 18t+2-6r\}, \\
2\{0, 6t-16-12r, 12t-8-6r\}, & 2\{0, 6t-17-12r, 18t-8-6r\}, \\
2\{0, 6t-14-12r, 12t-7-6r\}, & 2\{0, 6t-15-12r, 18t-7-6r\}, \\
2\{0, 6t-13-12r, 12t-4-6r\}, & 2\{0, 6t-11-12r, 18t-4-6r\}, \\
2\{0, 6t-10-12r, 12t-5-6r\}, & 2\{0, 6t-9-12r, 18t-5-6r\}, \\
2\{0, 6t-7-12r, 12t-3-6r\}, & 2\{0, 6t-8-12r, 18t-3-6r\},
\end{array}
$$

for $r \in [0, t/2-2]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is even,

$$
\begin{array}{lllll}
2\{0, 1, 9t+2\}, & 2\{0, 3, 9t+2\}, & 2\{0, 5, 9t+4\}, & 2\{0, 2, 15t+1\}, & 2\{0, 4, 15t+2\}, \\
2\{0, 2, 9t+3\}, & 2\{0, 5, 9t+3\}, & 2\{0, 1, 15t+4\}, & 2\{0, 3, 15t+8\}, & 2\{0, 4, 15t+7\}.
\end{array}
$$

□

Construction 19 *For $g \equiv 2 \pmod{12}$ and $g > 2$, there exists a $(6g, g, 3, 6)_6$-DF over Z_{6g}.*

Proof Let $g = 12t + 2$ where $t \geq 1$. The base blocks are

$$
\begin{array}{llll}
\{0, 1, 10\}, & 6\{0, 4, 18t+5\}, & \{0, 11, 30t+10\}, & 6\{0, 3, 30t+7\}, \\
\{0, 1, 11\}, & 6\{0, 5, 18t+4\}, & 4\{0, 10, 30t+9\}, & 6\{0, 12t+1, 24t+3\}, \\
4\{0, 1, 18t+3\}, & \{0, 7, 30t+9\}, & 5\{0, 7, 30t+8\}, & 2\{0, 18t-3, 36t+5\}, \\
4\{0, 9, 18t+7\}, & \{0, 8, 30t+9\}, & 5\{0, 8, 30t+10\}, & 2\{0, 18t-2, 36t+5\}, \\
4\{0, 11, 18t+8\}, & \{0, 9, 30t+8\}, & 6\{0, 2, 30t+5\}, & 2\{0, 18t+2, 36t+5\},
\end{array}
$$

and when $t \geq 2$,

$$
\begin{array}{ll}
6\{0, 12t-11-12r, 24t-3-6r\}, & 6\{0, 12t-10-12r, 36t-1-6r\}, \\
6\{0, 12t-8-12r, 24t-1-6r\}, & 6\{0, 12t-9-12r, 36t+1-6r\}, \\
6\{0, 12t-7-12r, 24t-2-6r\}, & 6\{0, 12t-5-12r, 36t+2-6r\}, \\
6\{0, 12t-3-12r, 24t+1-6r\}, & 6\{0, 12t-4-12r, 36t+4-6r\}, \\
6\{0, 12t-1-12r, 24t+2-6r\}, & 6\{0, 12t-2-12r, 36t+3-6r\},
\end{array}
$$

for $r \in [0, t-2]$. □

Construction 20 *For $g \equiv 5 \pmod{6}$, there exists a $(6g, g, 3, 12)_{12}$-DF over Z_{6g}.*

Proof Let $g = 6t + 5$ where $t \geq 0$. The base blocks are

$$
\begin{array}{llll}
\{0, 6t+1, 12t+8\}, & 3\{0, 6t+4, 12t+11\}, & 6\{0, 6t+4, 12t+13\}, & 2\{0, 12t+7, 24t+16\}, \\
2\{0, 6t+1, 12t+9\}, & 4\{0, 6t+2, 12t+7\}, & 6\{0, 6t+1, 18t+14\}, & \{0, 12t+8, 24t+16\}, \\
2\{0, 6t+2, 12t+9\}, & 4\{0, 6t+3, 12t+11\}, & 3\{0, 6t+1, 18t+15\}, & \{0, 12t+8, 24t+19\}, \\
2\{0, 6t+3, 12t+8\}, & 6\{0, 6t+2, 12t+5\}, & 3\{0, 6t+4, 18t+15\}, &
\end{array}
$$

and when $t \geq 2$,

$6\{0, 6t-10-12r, 12t-1-6r\}$, $6\{0, 6t-11-12r, 18t+9-6r\}$,
$6\{0, 6t-8-12r, 12t+2-6r\}$, $6\{0, 6t-9-12r, 18t+10-6r\}$,
$6\{0, 6t-7-12r, 12t+1-6r\}$, $6\{0, 6t-5-12r, 18t+11-6r\}$,
$6\{0, 6t-4-12r, 12t+3-6r\}$, $6\{0, 6t-3-12r, 18t+14-6r\}$,
$6\{0, 6t-1-12r, 12t+4-6r\}$, $6\{0, 6t-2-12r, 18t+13-6r\}$,
$6\{0, 6t-10-12r, 12t+5-6r\}$, $6\{0, 6t-11-12r, 18t+8-6r\}$,
$6\{0, 6t-9-12r, 12t+4-6r\}$, $6\{0, 6t-8-12r, 18t+9-6r\}$,
$6\{0, 6t-7-12r, 12t+7-6r\}$, $6\{0, 6t-5-12r, 18t+11-6r\}$,
$6\{0, 6t-3-12r, 12t+8-6r\}$, $6\{0, 6t-4-12r, 18t+10-6r\}$,
$6\{0, 6t-1-12r, 12t+9-6r\}$, $6\{0, 6t-2-12r, 18t+13-6r\}$,

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is odd,

$6\{0, 2, 9t+6\}$, $6\{0, 3, 9t+8\}$, $6\{0, 5, 9t+7\}$, $6\{0, 1, 15t+14\}$, $6\{0, 4, 15t+16\}$,
$6\{0, 2, 9t+12\}$, $6\{0, 4, 9t+11\}$, $6\{0, 1, 15t+13\}$, $6\{0, 3, 15t+17\}$, $6\{0, 5, 15t+16\}$.

□

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